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Dynamics of a string moving with time-varying speed

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Abstract

The paper is dedicated to mathematical analysis of the idealized model of a string moving at a variable speed. Resonance case is studied and it is shown that an infinite system of equations cannot be reduced. Some exact solutions are obtained.
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1. Introduction

Dynamics of a conveyor belt moving with a time-varying velocity, where the belt is modeled through a string, is analyzed in Refs. [1–4]. It is worth noting that the described approach is suitable for modeling of low system's frequencies (low modes). However, in order to consider higher modes properly, the bending belt stiffness should be accounted [2]. Notice that although the introduced and discussed string model presented in Ref. [1] does not represent real behavior for the higher order modes, it does represent this behavior for lower order modes.

The stated problem is mathematically interesting and it will be further studied in this report following the steps introduced in Ref. [1]. Since the bending conveyor belt stiffness is of zero value, the corresponding object is further referred to as a mathematical string (MS).

2. Formulation of the problem

Here we follow steps introduced in Ref. [1]. Let MS move in x direction (Fig. 1) at velocity $V(t)$. Differentiation of the displacement $U(x, t)$ yields the following formula:

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \frac{\partial U}{\partial t} + \frac{\partial U}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} = U_t + V(t)U_x,$$

and one more differentiation gives

$$\frac{\mathrm{d}^2 U}{\mathrm{d}t^2} = U_{tt} + 2VU_{xt} + V^2 U_{xx} + V_t U_x. \tag{1}$$

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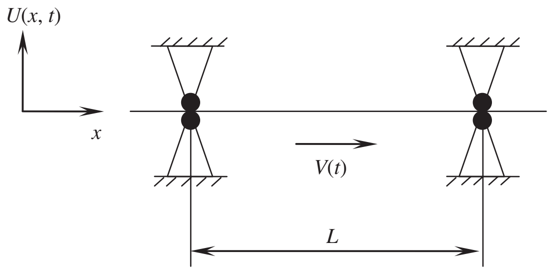

Fig. 1. Movement of MS.

MS motion is governed by the following equation:

$$TU_{xx} = \rho \frac{\mathrm{d}^2 U}{\mathrm{d}t^2}, \tag{2}$$

where T is the string tension and ρ is the mass density of the string; T and ρ are assumed to be constant.

Formulas (1) and (2) yield

$$c^2 U_{xx} = U_{tt} + 2VU_{xt} + V^2 U_{xx} + V_t U_x, \tag{3}$$

where $c = \sqrt{T/\rho}$ is the wave speed.

Assume that the MS ends are fixed in the vertical direction, i.e.

$$U = 0 \quad \text{for} \quad x = 0, L. \tag{4}$$

The following initial conditions are taken

$$U = f(x), \quad U_t = \varphi(x) \quad \text{for} \ t = 0. \tag{5}$$

Assuming that the MS velocity $V(t)$ is small in comparison to the sound velocity c, the following approximation holds

$$V(t) = \varepsilon(V_0 + \alpha \sin \omega t),$$

where $V_0 > 0$, $V_0 \geqslant |\alpha|$, $\varepsilon \ll 1$, V_0 is constant.

Finally, the MS motion is governed by the following equation

$$c^2 U_{xx} - U_{tt} = \varepsilon[\alpha\omega \cos(\omega t) U_x + 2(V_0 + \alpha \sin \omega t) U_{xt}] + \varepsilon^2 (V_0 + \alpha \sin \omega t)^2 U_{xx}. \tag{6}$$

For the sake of simplicity, we strongly simplify Eq. (6), assuming $c = \alpha = V_0 = 1$. Besides, in order to study the simplest resonance case we take $L = \pi$, $\omega = 1$.

Below, the problem is reduced to investigation of the following equation

$$U_{xx} - U_{tt} = \varepsilon[U_x \cos t + 2(1 + \sin t) U_{xt}] + \varepsilon^2 (1 + \sin t)^2 U_{xx}, \tag{7}$$

$$U = 0 \quad \text{for} \quad x = 0, \pi. \tag{8}$$

The following solution to Eq. (7) satisfying the boundary conditions (8) is applied

$$U(x, t) = \sum_{n=1}^{\infty} U_n(t) \sin nx. \tag{9}$$

Substituting Eq. (9) into Eq. (7) and splitting with respect to $\sin nx$, the following system of coupled ODEs is obtained:

$$\frac{\mathrm{d}^2 U_k}{\mathrm{d}t^2} + k^2 U_k = \varepsilon \sum \frac{2n}{2j+1} \left[U_n \cos t + 2(1 + \sin t) \frac{\mathrm{d}U_n}{\mathrm{d}t} \right] + \varepsilon^2 (1 + \sin t)^2 k^2 U_k, \tag{10}$$

where

$$\sum = \sum_{k=n-2j-1} - \sum_{k=2j+1+n} - \sum_{k=2j+1-n}, \quad k = 1, 2, \ldots .$$

In order to solve system (10) the multiple scale method is applied. Namely, introducing the variables $\tau = \varepsilon t$ and t, we take $U_k(t, \varepsilon) = V_k(t, \tau, \varepsilon)$ and hence

$$\frac{\mathrm{d}U_k}{\mathrm{d}t} = \frac{\partial V_k}{\partial t} + \varepsilon \frac{\partial V_k}{\partial \tau},$$
$$\frac{\mathrm{d}^2 U_k}{\mathrm{d}t^2} = \frac{\partial^2 V_k}{\partial t^2} + 2\varepsilon \frac{\partial^2 V_k}{\partial t \partial \tau} + \varepsilon^2 \frac{\partial^2 V_k}{\partial \tau^2}. \tag{11}$$

The functions $V_k(t, \tau, \varepsilon)$ are sought in the following form:

$$V_k(t, \tau, \varepsilon) = V_k^{(0)}(t, \tau) + \varepsilon V_k^{(1)}(t, \tau) + \varepsilon^2 V_k^{(2)}(t, \tau) + \ldots . \tag{12}$$

After substitution of Eqs. (11) and (12) into Eq. (10), and after splitting with respect to ε, the following equations are obtained:

$$\frac{\mathrm{d}^2 V_k^{(0)}}{\mathrm{d}t^2} + k^2 V_k^{(0)} = 0, \tag{13}$$

$$\frac{\partial^2 V_k^{(1)}}{\partial t^2} + k^2 V_k^{(1)} = -2 \frac{\partial^2 V_k^{(0)}}{\partial t\, \partial \tau} + \sum \frac{2n}{2j+1} \left[V_k^{(0)} \cos t + 2(1 + \sin t) \frac{\partial^2 V_k^{(0)}}{\partial t^2} \right], \tag{14}$$

....

The solution of Eq. (13) reads

$$V_k^{(0)} = A_k(\tau) \cos kt + B_k(\tau) \sin kt. \tag{15}$$

Since functions $A_k(\tau)$, $B_k(\tau)$ are defined through the lack of secular terms in Eq. (14), the following relations are obtained:

$$\frac{\mathrm{d}A_k}{\mathrm{d}\tau} = (k+1)B_{k+1} + (k-1)B_{k-1},$$
$$\frac{\mathrm{d}B_k}{\mathrm{d}\tau} = -(k+1)A_{k+1} - (k-1)A_{k-1}. \tag{16}$$

Note that so far Eqs. (1)–(16) are taken from paper [1].

3. Analysis of infinite systems

Assuming the solution of system (16) in the form

$$A_k(\tau) = C_k \mathrm{e}^{\lambda\tau}, \quad B_k(\tau) = D_k \mathrm{e}^{\lambda\tau}, \tag{17}$$

where λ, C_k, D_k are the constants, the following infinite system of linear algebraic equations is obtained:

$$\lambda C_k = (k+1)D_{k+1} + (k-1)D_{k-1},$$
$$\lambda D_k = -(k+1)C_{k+1} - (k-1)C_{k-1}, \quad k = 1, 2, \ldots . \tag{18}$$

Consider first the case $\lambda = 0$, and hence the system (18) yields

$$D_{2k} = C_{2k} = 0,$$
$$D_{2k+1} = (-1)^k \frac{D_1}{2k+1},$$
$$C_{2k+1} = (-1)^k \frac{C_1}{2k+1}, \quad k = 1, 2, \ldots . \tag{19}$$

Now let $\lambda \neq 0$. The system (18) can be reduced to the following form:

$$\lambda D_k = -(k+1)C_{k+1} - (k-1)C_{k-1}, \tag{20}$$

$$-\lambda^2 C_k = (k-1)(k-2)C_{k-2} + 2k^2 C_k + (k+1)(k+2)C_{k+2}. \tag{21}$$

Observe that Eq. (21) can be studied separately for C_{2k} and C_{2k+1}. Let us analyze Eq. (21) for C_{2k+1}, since for C_{2k} the results are analogous. For this case, the determinant of system (21) has the form:

$$\begin{vmatrix} 1+\dfrac{\lambda^2}{2} & 3 & 0 & 0 & 0 & \dots \\ \dfrac{1}{9} & 1+\dfrac{\lambda^2}{2\times 3^2} & 1\dfrac{1}{9} & 0 & 0 & \dots \\ 0 & \dfrac{6}{25} & 1+\dfrac{\lambda^2}{2\times 5^2} & \dfrac{21}{25} & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix} = 0. \tag{22}$$

Determinant (22) has a triple diagonal structure, and the terms on diagonals have the form

$$\frac{1}{2} - \frac{3}{2k} + \frac{1}{k^2}, \qquad 1 + \frac{\lambda^2}{2k^2}, \qquad \frac{1}{2} + \frac{3}{2k} + \frac{1}{k^2}. \tag{23}$$

Next, we are going to check if the infinite determinant (22) can be truncated. For this purpose the following inequalities should be verified [5,6]. Notice that if both Koch's conditions for reduction of infinite determinant of the form:

$$\sum_{k=1}^{\infty} |a_{kk} - 1| < \infty, \tag{24}$$

$$\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |a_{kj}|^2 < \infty, \quad k \neq j, \tag{25}$$

are satisfied, then the system can be truncated (a_{kj} are the determinant elements).

Although condition (24) is satisfied, condition (25) is not satisfied for determinant (22). In other words, system (18) cannot be truncated. This result has been obtained in Ref. [1] on the basis of numerical calculations.

4. Particular solution

Let us analyze relations (23). Note that a sum of terms in Eq. (23) is equal to $1 + (2/k^2)$, and for $\lambda^2 = 4$ the second term in Eq. (23) is equal to the sum of the first and third terms in Eq. (23) for any k.

Assuming

$$C_{2k+1} = (-1)^k C_1, \quad k = 1, 2, 3, \dots, \tag{26}$$

and analogously

$$C_{2k} = (-1)^{k-1} C_2, \quad k = 2, 3, 4, \dots, \tag{27}$$

instead of infinite systems of equations (18) one gets an infinite set of identical equations.

Thus, for two particular cases, $\lambda = 0$ and $\lambda^2 = 4$, an infinite system of equations has a solution. This observation enables construction of particular solutions for given initial conditions.

Namely, let us apply the initial conditions (5) in the following form:

$$U = A \sum_{j=1,3,5,\dots} (-1)^{(j-1)/2} \frac{1}{j} \cos jx, \qquad U_t = 0, \quad \text{for} \quad t = 0, \tag{28}$$

where A is constant.

Then, $D_k = 0$, $C_{2k+1} = A(-1)^k(1/2k+1)$, and up to accuracy of ε, one gets the following solution to problem (7), (8) in the resonance case:

$$U = A \sum_{j=1,3,5,\ldots} (-1)^{(j-1)/2} \frac{1}{j} \cos jt \cos jx. \tag{29}$$

The series (29) is convergent, and the solution (up to accuracy of ε^2 order) does not include the resonance terms. The solution for non-resonance terms reads

$$U = \varepsilon \sum_{j=1}^{\infty} (C_j^{(1)} \cos jt + D_j^{(1)} \sin jt) \cos jx,$$

where $C_j^{(1)}$, $D_j^{(1)}$ are certain coefficients.

Solution (29) can be improved to reach accuracy of ε^2, when the initial conditions (28) are modified, i.e.

$$U = A \sum_{j=1,3,5,\ldots} (-1)^{(j-1)/2} \frac{\cos jx}{j} + \varepsilon \sum_{j=1}^{\infty} C_j^{(1)} \cos jx, \tag{30}$$

$$U_t = \varepsilon \sum_{j=1}^{\infty} j D_j^{(1)} \cos jx. \tag{31}$$

If the series (30), (31) are convergent, then the obtained solution has the accuracy of ε^2.

5. Why is the mathematical string model incorrect?

As it has been already shown, the considered mathematical string model yields rather strange results. It is, however, clear from the mathematical point of view since this model is valid only for a few first modes. In order to estimate a possible application domain of this model we take $V = 0$ and we consider more general beam model [1,2] governed by the following equations:

$$U_{tt} - c^2 U_{xx} + b^2 U_{xxxx} = 0, \tag{32}$$

$$U = U_{xx} = 0 \quad \text{for} \quad x = 0, L. \tag{33}$$

In the above $b^2 = EI/\rho$, E is Young's modulus, I is the moment of inertia with respect to the horizontal axis.

We are going to find the frequency of vibrations in the problem (32), (33) applying the formula

$$U = C e^{i\omega t} \sin nx, \quad n = 1, 2, \ldots$$

As a result one gets

$$\omega^2 = n^2 \left[c^2 + b^2 n^2\right].$$

Note that the second term in brackets can be neglected when $n < n^* = c/b$.

The inequality $n < n^*$ defines the application interval of the string model. However, for $n > n^*$ the full equation (5) should be studied.

The following main conclusion is drawn from this analysis. Namely, the string model represents the asymptotic approximation of the full equation (5), and can be applied only for n^* first vibration modes. Furthermore, an application of the string-model for all vibration modes is wrong and does not have any physical meaning.

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